International Journal of Theoretical Physics, Vol. 39, No. 4, 2000

Dirac Quantization in Kantowski–Sachs Spacetime

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Received March 5, 1999

By forming the square root of the Wheeler–DeWitt equation and applying it to a minisuperspace composed of a Kantowski–Sachs universe, we derive a cosmological wave function with conserved current and positive-definite probability density.

1. INTRODUCTION

Since Hartle and Hawking (1983) and Vilenkin (1982, 1988) explored the birth of the universe in the frame of quantum gravitation, quantum cosmology has gone through three main phases: (1) quantum theory of a single universe (Halliwell, 1988), (2) wormhole mechanism and topological variation of space (Coleman, 1988; Klebanov *et al.*, 1989), and (3) quantum theory of multiple universes, third quantization, i.e., universal quantum field theory (Giddings and Strominger, 1989; Hosoya and Morikawa, 1989; Duncan, 1990; Peley, 1991; Garay, 1993; Shen, 1995). Clearly, investigating the properties of the universe during this period (about 10^{-43} s) is very important for a complete understanding of the universe.

The universal wave function in the model of a closed Friedmann universe with scalar field is given by Hartle and Hawking (1983) through a Wheeler–DeWitt equation. However, the Wheeler–DeWitt equation is a hyperbolic equation, so the universal wave function obtained in this method is ambiguous in the same physical sense as the Klein–Gordon equation in quantum mechanics.

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0020-7748/00/0400-1049$18.00/0

In order to handle the crucial difficulty in the density interpretation of the wave function in quantum cosmology, D'Eath *et al.* (1993) investigated in detail a supersymmetric Bianchi model employing the root square of the Wheeler–DeWitt equation. By means of this method, the Friedmann universe with charged electric scalar field was studied by Malle (1995). This work has shown that the probability interpretation is reliable.

In this paper we apply the square root of Wheeler–DeWitt equation to a minisuperspace composed of a Kantowski–Sachs universe, and deduce a wave function with conserved current and positive-definite probability density.

2. DIRAC QUANTIZATION

The metric in a Kantowski–Sachs universe (Kantowski and Sachs, 1966) reads

$$ds^2 = \frac{G}{2\pi}[-N^2(t)dt^2 + a^2(t)dr^2 + b^2(t)\,d\Omega_2^2] \tag{1}$$

where G is the Newton constant and $N(t)$ is the lapse function; the coordinate r is taken to be periodic with a period of 2π. Here $d\Omega_2^2$ is the unit S^2 sphere, and the topological structure of this metric is $R^1 \otimes S^1 \otimes S^2$.

We shall consider an Einstein action of coupled real scalar field $\Phi = \varphi/\sqrt{4\pi G}$,

$$S = \int dt\,\frac{N}{2}\left[-\frac{1}{N^2}(a\dot{b}^2 + 2b\dot{a}\dot{b}) - a + \frac{ab^2}{N^2}\dot{\varphi}^2\right] \tag{2}$$

Setting

$$c = ab \tag{3}$$

and substituting Eq. (3) into Eq. (2), we have

$$S = \int dt\left[-\frac{\dot{c}^2}{2Na} + \frac{c^2\dot{a}^2}{2Na^3} - \frac{Na}{2} + \frac{c^2}{2Na}\dot{\varphi}^2\right] \tag{4}$$

Equation (4) yields the Hamiltonian of the system

$$H = \frac{Na}{2c^2}(a^2\Pi_a^2 - c^2\Pi_c^2 + c^2 + \Pi_\varphi^2) \tag{5}$$

Taking a classical constraint and operator transformation, we obtain the Wheeler–DeWitt equation in the form

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \varphi^2} + e^{2x}\right)\Psi(x, y, \varphi) = 0 \tag{6}$$

where $x = \ln c$, $y = \ln a$, and we make the simplest choice of factor-ordering ambiguity $p = 1$ (Hosoya and Morikawa, 1989).

We form the square root of the Wheeler–DeWitt equation (D'Eath *et al.*, 1993; Malle, 1995) because the wave equation in minisuperspace (x, y, φ) is required to be linear. Assuming x to be 'time' variable and y, φ to be 'space' variables, we introduce the wave equation of the form

$$i\,\frac{\partial\Psi}{\partial\Omega} = -i\left(\sigma_y\,\frac{\partial\Psi}{\partial y} + \sigma_\varphi\,\frac{\partial\Psi}{\partial\varphi}\right) - iW(x)\Psi \tag{7}$$

where $W(x)$ is real function of x and

$$\sigma_y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{8}$$

$$\sigma_\varphi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{9}$$

$$\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{10}$$

Substituting Eqs. (8)–(10) into wave equation (7), we obtain the matrix equation

$$\begin{bmatrix} i\,\dfrac{\partial\Psi_1}{\partial x} \\[2ex] i\,\dfrac{\partial\Psi_2}{\partial x} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial\Psi_2}{\partial y} + \dfrac{1}{i}\,\dfrac{\partial\Psi_2}{\partial\varphi} + \dfrac{1}{i}\,W(x)\Psi_1 \\[2ex] -\dfrac{\partial\Psi_1}{\partial y} + \dfrac{1}{i}\,\dfrac{\partial\Psi_1}{\partial\varphi} + \dfrac{1}{i}\,W(x)\Psi_2 \end{bmatrix} \tag{11}$$

where Ψ_α ($\alpha = 1, 2$) must satisfy the Wheeler–DeWitt equation (6),

$$\frac{\partial^2\Psi_\alpha}{\partial x^2} = \frac{\partial^2\Psi_\alpha}{\partial y^2} + \frac{\partial^2\Psi_\alpha}{\partial\varphi^2} - 2e^{2x}\,\Psi_\alpha \tag{12}$$

From Eq. (11), Eq. (12) can reduces to

$$\frac{\partial^2\Psi_\alpha}{\partial x^2} = \frac{\partial^2\Psi_\alpha}{\partial y^2} + \frac{\partial^2\Psi_\alpha}{\partial\varphi^2} - 2W\,\frac{\partial\Psi_\alpha}{\partial x} - W^2\Psi_\alpha - \frac{\partial W}{\partial x}\,\Psi_\alpha \tag{13}$$

Taking into account Eqs. (12) and (13), we can constrain $W(x)$ by the following equation:

$$\frac{dW}{dx} + 2fW + W^2 = 2e^{2x} \tag{14}$$

where

$$f = \frac{\partial \ln \Psi_\alpha}{\partial \Psi} \tag{15}$$

Equation (14) takes the form of a Riccati equation. By means of Eq. (11), the conserved current corresponding to Eq. (7) can be expressed as

$$i\,\frac{\partial \Psi}{\partial x} = H_\sigma\,\Psi \tag{16}$$

Taking the complex conjugate to Eq. (16), we easily get

$$-i\,\frac{\partial \tilde{\Psi}^*}{\partial x} = (\tilde{H}_\sigma\,\Psi)^* \tag{17}$$

By combining Eq. (16) with (17), we find

$$i\tilde{\Psi}^*\,\frac{\partial \Psi}{\partial x} + i\,\frac{\partial \tilde{\Psi}^*}{\partial x}\,\Psi = \tilde{\Psi}^* H_\sigma \Psi - (\tilde{H}_\sigma \Psi)^* \Psi \tag{18}$$

Equation (18) can be rewritten as

$$\frac{\partial \rho}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_\varphi}{\partial \varphi} = 0 \tag{19}$$

where ρ, j_y, and j_φ are respectively given by

$$\rho = k\exp\left(\int W\,dx\right)\tilde{\Psi}^*\Psi \tag{20}$$

$$j_y = k\exp\left(\int W\,dx\right)\tilde{\Psi}^*\sigma_y\Psi \tag{21}$$

$$j_\varphi = k\exp\left(\int W\,dx\right)\tilde{\Psi}^*\sigma_\varphi\Psi \tag{22}$$

and k is a positive real constant.

Therefore, Eqs. (19)–(22) reveal that the wave equation for two-component cosmological wave function satisfies the conserved current with positive-definite density ρ, where the potential is a definite factor.

The analyses above suggest that, by exploiting the square root of the Wheeler–DeWitt equation, the positive-definite probability density and conserved current required in quantum mechanics can be satisfied.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China under Grant No. 19873013.

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